Entropic Characterisation of Diffusion*

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The thermodynamic approach is applied for the description of normal and anomalous diffusion of one-dimensional maps on a grid of unit cells. The characteristic entropy functions are calculated. For the anomalous cases, the locations of the critical lines are determined.

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A diffusion-related free energy has been used recently [1–3] for the characterisation of diffusional behaviour [4–10]. In [2–3], this function has been evaluated numerically for typical model systems. However, in some sense the associated entropy function is able to give a more condensed picture of the system under investigation [11, 12]. For a special class of chaotic systems related to a different problem if compared to the present context, the entropy representation has already been investigated in detail [12–14]. The present application of this general concept to diffusion, however, is rather new [1, 15]. The anomalous enhanced and dispersive one-dimensional diffusion arises from maps where an intermittent fixed point leads to orbits which are connected to an anomalous amount of or anomalous absence of transport. Though unstable, the probability to stay on such an orbit decreases not exponentially but decays algebraically. Therefore, the fluctuations may not behave according to the Gaussian limit theorem but may follow Lévy's stable distributions [16]. In the reduced map, in this way the phenomenon of a phase transition is generated, which persists in the $f(\alpha)$-, in the Renyi entropies- and in the Lyapunov exponent spectrum [17, 18]. For the transport properties, a side-effect may be that we have no longer the normal diffusive behaviour which is characteristic for Gaussian fluctuations (whether this occurs depends on the size of the exponent of intermittency [1]). More precisely, the second moment may behave as

$$\langle r^2(t)\rangle \sim t^{\alpha} \tag{1}$$

with $\alpha > 1$ or $\alpha < 1$. This is when we speak of enhanced or of dispersive diffusion, respectively. Of special importance is the diffusion behaviour of Hamiltonian systems. In these systems, areas of enhanced, of dispersive, and of normal diffusional behaviour may coexist [19–27].

In our treatment, let us consider the most significant model systems without constructing the map explicitly. To model an enhanced diffusive system, we start from a suitable reduced map with a symbolic partition of a sufficiently high level. This partition can be furnished by a diffusive map on a grid of unit cells with generating intermittent branch $f(x) = (1-\varepsilon)\,x + \bar{a}\,x^{\bar{z}}, 0 < x < 1/2$, where the parameter $\bar{a}$ is chosen to be $\bar{a} = 2^{\bar{z}}(1-\varepsilon/2)$. For $\varepsilon = 0$, a marginally stable fixed point arises. A criterion which allows one to obtain an enhanced diffusional system is to demand that in the associated map all partition elements with exception of the first induce a jump to a neighbouring unit cell in the enhanced case, whereas only the first element leads to a jump in the dispersive case. For simplicity we suppose that the monotonous pieces in the partition are of linear shape. In order to induce an algebraic decay from the element which includes the marginally stable fixed point, our partition is equipped with the following (idealised) property [28]: We require the partition elements to scale as

$$w_k = 0.5\,(k^{-(1/(\bar{z}-1))} - (k+1)^{-(1/(\bar{z}-1))}), \tag{2}$$

where $\bar{z}$ is the exponent of intermittency. In the present system, the size of the partition elements is at the same time a measure for the dynamical stability of a chain of k steps, where k is the index of the partition element. Explicitly, we have

$$\log_e w_k = -\,k\,\lambda(k), \tag{3}$$

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where $\lambda(k)$ denotes the Lyapunov exponent of k steps. Furthermore the k-step Lyapunov exponents are a measure for the probability $P_{1,k}$ to land on the element $A_k(C_k)$ after having been reinjected at the elements with index 1 [28]. Therefore, in this system an extremely simple relation between the dynamical and the probabilistic measures results. Then, the collection of the fundamental periodic orbits is an easy task. For the orbits of length k the stability exponents are readily given by the size of the partition element which is associated with the starting place of the orbit, after the reinjection process has taken place. The curvature terms [29] cancel in the linearised case. In order to obtain the diffusion-related probabilities, note that in the symbolic partition, to each partition element the ability to induce k_i jumps of length i can then the attached as a signed measure. In this way, diffusion-related probabilities

$$p(k)^q = \mathrm{e}^{\pm q(k-1)/k} \tag{4}$$

(enhanced case), and

$$p(k)^q = e^{\pm q/k} \tag{5}$$

(dispersive case) are obtained, where the signs indicate the direction of the jumps.

Let us now shortly explain how the thermodynamic treatment of this problem works. First, the generalised, diffusion-related free energy F_d of the system is calculated [1–3, 12–15, 17, 18]. In accordance with the properties of the model and in order to take advantage of the fact that the summation in the partition sum can be restricted to all fundamental orbits of different lengths k, we consider the grandcanonical partition function. From the generalised free energy, the associated entropy function S, finally, is obtained from F_d with the help of the Legendre transformation [16]

$$S(v, \varepsilon) = F_\mathrm{d}(q, \beta) - \frac{\partial F_\mathrm{d}(q, \beta)}{\partial \beta}\beta + \frac{\partial F_\mathrm{d}(q, \beta)}{\partial q} q, \tag{6}$$

where v and ε are the new variables which are conjugated to q and β. As for the usual generalised entropy function, this relationship can be derived from the characteristic relation between the number N of orbits with the same value of the scaling variables $v(q)$ and $\varepsilon(\beta)$ and the entropy function. In this way, the diffusion-related entropy function is expressed in the variables ε (the dynamical stability exponents or length scales) and v (the velocity on the grid). The change from the free energy to the entropy function is by far not a trivial one, if the free energy is given as a result from numerical calculations. The numerical inaccuracy is then transformed from a tiny region around the Sinai-Bowen-Ruelle (SBR)-point (which is obtained for the parameters $\beta = 1$, $q = 0$) to a more extended size which is more adequate to the importance of this region.

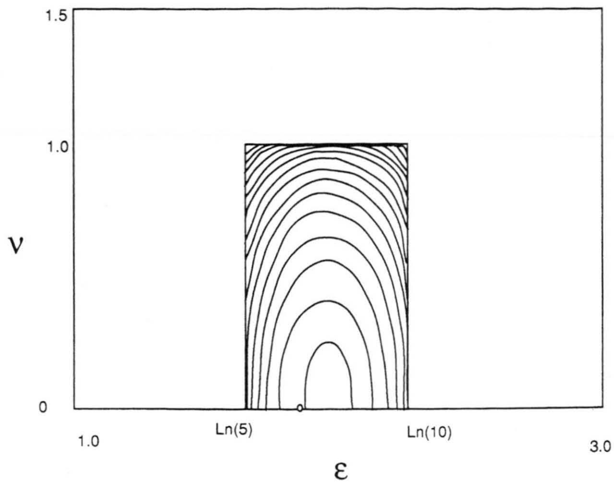

Fig. 1. Diffusion-related entropy function of the asymmetric drift-free tent map as the prototype of a hyperbolic map with a general fluctuation behaviour displaying normal diffusion. The contour lines are shown, decreasing from the value 1.9 in steps of 0.1. Because of the symmetry with respect to the sign of v, only the part of the entropy function which corresponds to positive velocities is shown. In all Figures the circle indicates the location of the SBR-point (see text).

As the first example serves the diffusive drift-free asymmetric tent map, for which the free energy can easily be calculated. In comparison with the simpler system treated in [1, 7, 15], the support of the generalised entropy function $S(\varepsilon, v)$ is no longer restricted to one single length-scale (see Figure 1). As a consequence of the hyperbolicity of this model, we do not obtain nonanalyticities. In accordance, the numerically calculated associated entropy function leads to strictly convex specific entropy functions. As is usual, e.g. [12–14], the maximum of this function reflects the topological entropy ($\log_e(7)$). The system leads to normal diffusion, where the diffusion coefficient D [1, 2, 19] (i.e., the leading constant if $\alpha = 1$) can be calculated in the thermodynamic formalism by making use of the relation

$$D = \frac{1}{2}\frac{\partial^2}{\partial q^2} F_\mathrm{d}(q, \beta)\,|_{q=0,\ \beta=1}\,. \tag{7}$$

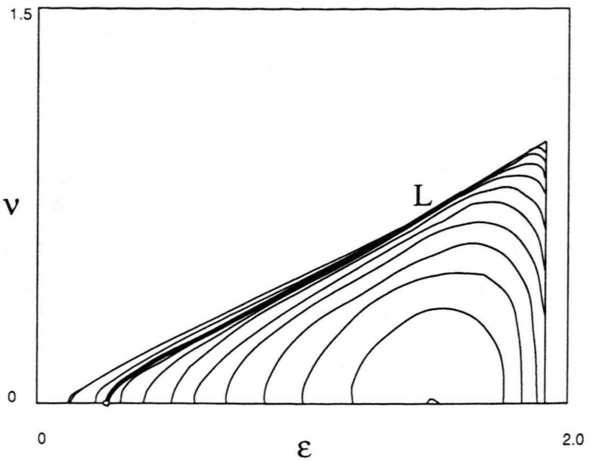

Fig. 2. Diffusion-related entropy function for the dispersive map for $\bar{z} = 3.0$. The contour lines are shown, decreasing from the value 1.0 to 0 in steps of 0.1. The top contour line corresponds to the value 1.09. The figure shows the approximation obtained from a partition function of the level $n = 55$. The corresponding critical line is denoted by L.

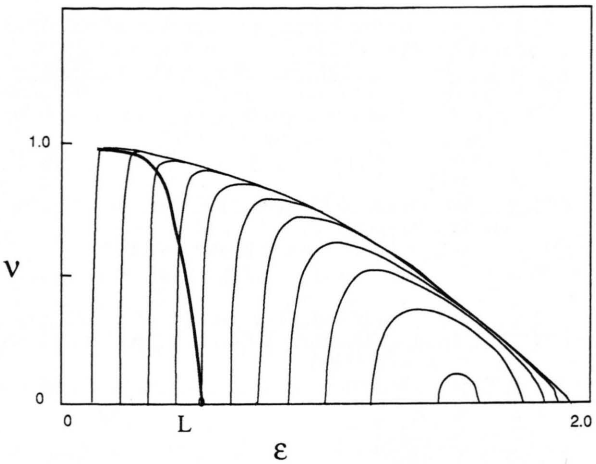

Fig. 3. Diffusion-related entropy function for the enhanced map ($\bar{z} = 3.0$) showing contour lines decreasing from the value 1.0 to 0 in steps of 0.1. The top contour line corresponds to the value 1.09. The figure shows the approximation obtained from a partition function of the level $n = 35$.

Due to the relation with the probability, the length scale $\varepsilon = 0$ is an indicator for the dominant diffusional property of the system. In the enhanced system this length scale is associated with the velocity $v = 1$, whereas it is associated with the velocity $v = 0$ in the dispersive case (see Figures 2–3). Of special interest is the diffusion coefficient which is again calculated by using the behaviour of the free energy at the SBR-point. In rough terms, anomalous diffusion is obtained if D vanishes (dispersive, sublinear diffusion) or if it diverges (accelerated, superlinear diffusion). For these simple nonhyperbolic models, the diffusion coefficients can be calculated explicitly [1–3]. In this way, the anomaly in the diffusional behaviour can be shown for $\bar{z} > 3/2$, for the enhanced case and for $\bar{z} > 2$ in the dispersive case (where $\bar{z}$ denotes the exponent of intermittency). Whereas for the dispersive case the condition for anomaly is easy to understand, for the enhanced case this is far more difficult (this fact corresponds to the more complicated picture in the random-walk approach [35]). It is no surprise that the ability to induce anomalous diffusion is determined by the behaviour of the free energy at the associated critical line [1–3, 17, 18] at the SBR-point. For more information on the nature of this dependence we refer to [1–3], where the connection with the phenomenon of phase transitions [30–34] is discussed.

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